Title: Fun with Fractals

Link to Outcomes:

- **Communication** Students will demonstrate the ability to communicate through mathematics and technology by working in groups and writing conjectures.

- **Reasoning** Students will demonstrate their ability to reason inductively and deductively by making conjectures as to the outcomes of an iterative process.

- **Connections** Students will discover connections between the iterative process of fractals and the real world.

- **Algebra/Geometry** Students will demonstrate their ability to transform geometric figures by calculating and applying transformation matrices.

- **Discrete Math** Students will represent problem situations using matrices. They also will develop and analyze algorithms dealing with patterns found in Pascal's Triangle and the Koch Curve.

- **Technology** Students will use a graphing calculator to explore matrices and fractals. They will also enter a program that creates a fractal.

Brief Overview:

This unit explores relationships and patterns with numbers and geometric figures. Pascal's Triangle is used to introduce Sierpinski's Triangle. Properties of Koch's Curve are explored. Matrices are employed in the construction of Sierpinski's Triangle.

Grade/Level:

Grades 9 - 12; Algebra I through Pre-Calculus

Duration/Length:

This unit is expected to take 3 - 5 class periods depending on the level of the class.

Prerequisite Knowledge:

- Concepts of measurement and perimeter
- Cooperative group processes
- Knowledge of trigonometric functions (Day 4 and Extension activities only)

Objectives:

Students will be able to:

- give a basic definition of "fractals."
- use Pascal's Triangle to obtain Sierpinski's Triangle.
- define and recognize "iteration."
- use an iterative technique to form a fractal.
- use matrices to construct Sierpinski's Triangle (for Trigonometry classes and above).
- enter a Sierpinski Triangle program into a graphing calculator (optional).

Materials/Resources/Printed Materials:

- Student Worksheets (Student Resources #1-10)
- Teacher Resources #1-5
- Colored pens/pencils (optional)
- Overhead transparency paper and markers (one for every pair of students)
- Ruler (one for every pair of students)
- Die (one for every pair of students)
- Graph paper (for Trigonometry classes and above)
- Graphing Calculator – TI-81, TI-82 or Casio (optional)

Development/Procedures:

Day 1:

As a warm-up exercise, students are asked to complete a portion of Pascal's Triangle (Student Resource #1). This is used to color in the odd numbers whereby Sierpinski's Triangle will emerge (Student Resource #2). Students are required to find the pattern and describe it in words. Teacher should give background information on fractals (Teacher Resource #1), and hand out vocabulary lists (Student Resource #3 and Teacher Resource #2) to be filled out as appropriate. As an extension, students may color other patterns (Optional Worksheet, Student Resource #4) such as multiples of 3 or 7, to make Sierpinski-like fractals.

Day 2:

Students working in pairs play "The Chaos Game" (Student Resource #5 and Teacher Resource #3). Forming groups of four, they conjecture what the picture will look like if 1000 points were plotted. They also conjecture what effect choosing a point other than one of the vertices to begin might have. (Optional: Students can enter the program (Student Resource #6) on a graphing calculator to further explore the Chaos Game.)

Day 3:

Students will use their knowledge of iteration to create 4 stages of the Koch Curve (Student Resource #7 and Teacher Resource #4). They will be asked to give the perimeter of each of their figures and conjecture what the perimeter will be at the *nth* stage.

Day 4 (for Trigonometry classes and above):

Students will explore the effects of scaling, rotations, and translations on geometric figures by calculating and applying transformation matrices (Student Resource #8 and Teacher Resource #5). If graphing calculators are available, students will be able to examine how the Chaos program uses matrices to produce Sierpinski's Triangle (Student Resource #9).

Evaluation:

Students are expected to compile a small portfolio of worksheets, notes, constructions, definitions, and an optional student "Unit Evaluation" (Student Resource #10).

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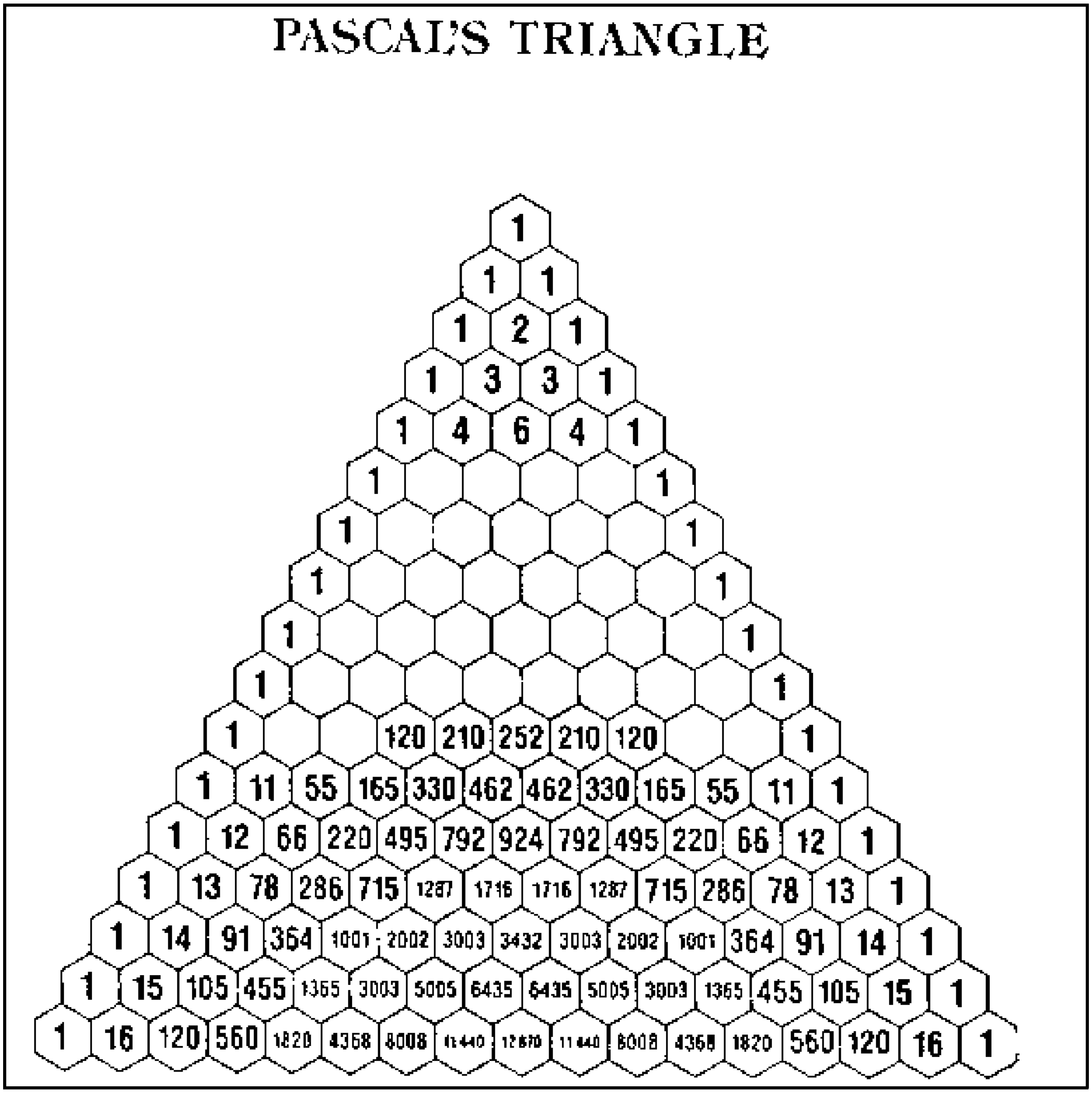
Student Resource #1

Worksheet 1

Name__________________

Fill in the blanks in Pascal's Triangle.



Student Resource #2

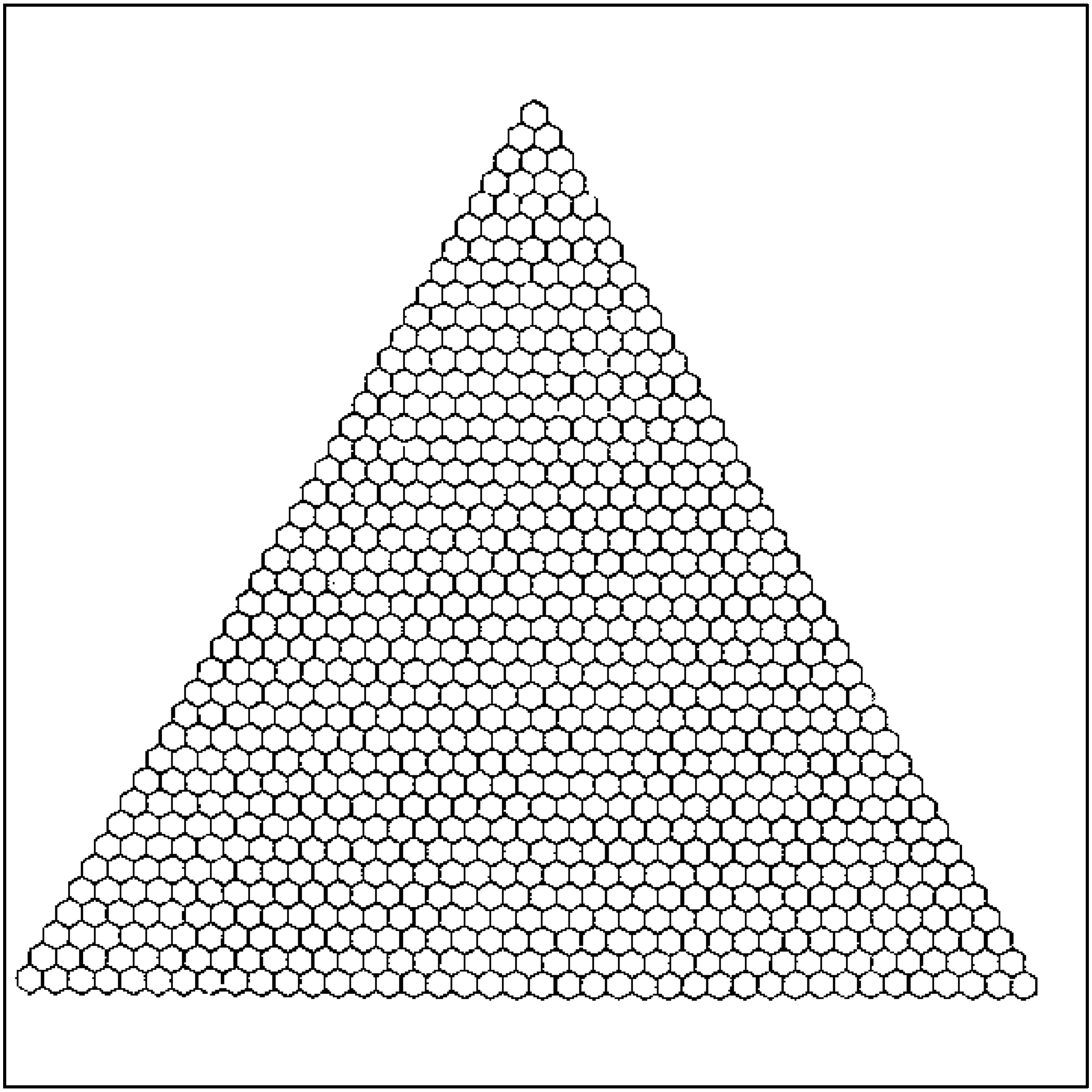
Worksheet 2

Name____________________

Patterns in Pascal's Triangle

Use Pascal's Triangle worksheet and color in the odd numbered hexagons. Find a pattern to color a hexagon based on the two hexagons above it. Write your pattern here. ______________
____

Color this triangle according to your pattern.

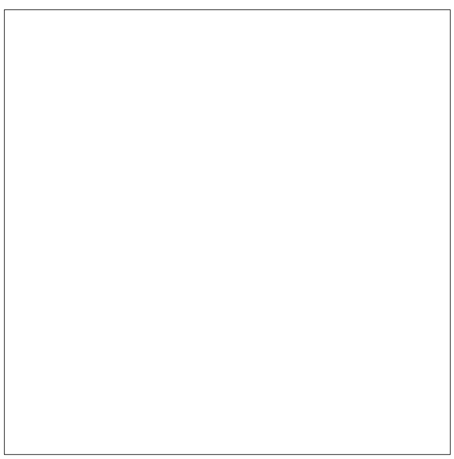


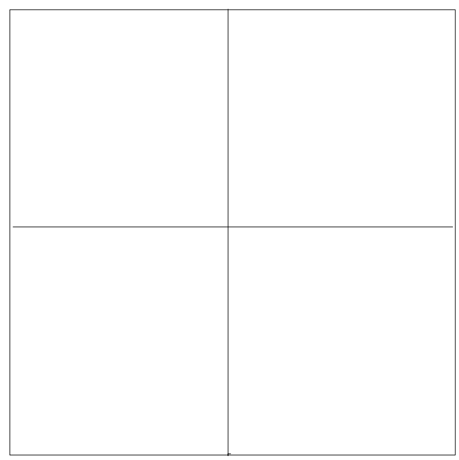
Teacher Resource #1

Notes on Fractals

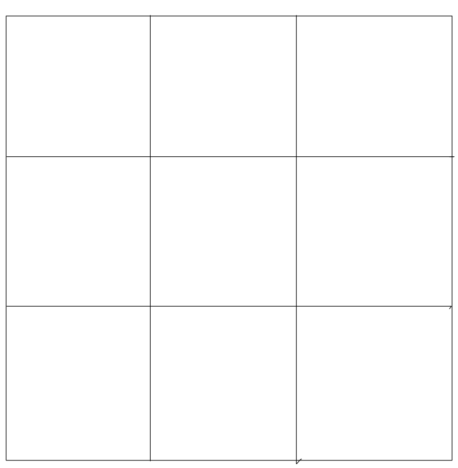
A fractal is an object which exhibits three main characteristics: iteration, self-similarity, and non-integral fractal dimension. Iteration requires that the figure be formed by repeating the same process, whatever it be, over and over again. Self-similarity can be seen in many facets of nature. Think of a tree. It consists of large branches which consist of smaller branches, which consist of smaller twigs, all of which taken by themselves form a reasonably good model of a tree.

Fractal dimension can be measured in several ways. In order for students to calculate fractal dimension, they must know basic logarithms or exponents. Fractal dimension can be calculated by using the formula $a=s^{-D}$ where a is the number of pieces, s is the reduction factor, and D is the fractal dimension. To find the dimension of a rectangle, notice that if we reduce it by a factor of ½, it takes four of these reductions to cover the original rectangle.





If we apply the formula to this data we have $4=1/2^{-D}$ or $2^D=4$, so a rectangle has dimension 2. We can also reduce the rectangle by 1/3, taking 9 rectangles to cover the original rectangle.



Using the formula with this data, we have $3^D=9$, showing again a dimension of 2.

Student Resource #3

Name: ________________

Vocabulary List - Fun with Fractals

Define the following in your own words:

Pascal’s Triangle

Fractal

- Self-similar
- Dimension
- Iteration

Sierpinski Triangle

Transformation

- Scaling
- Rotation
- Translation

Vocabulary List - Fun with Fractals
Answer Sheet

Define the following in your own words:

Pascal's Triangle - a triangular arrangement of numbers each obtained by adding the two above it

Fractal - a figure with the following characteristics:

Self-similar -	parts of a figure contain small replicas of the whole
Dimension -	a means of measuring the complexity of objects
Iteration -	a repetitive process whereby one answer is used to find the next
Sierpinski Triangle -	a fractal consisting of a large triangle, the center of which is removed, leaving the three corner triangles. This process is repeatedly iterated. (Students may draw a diagram.)

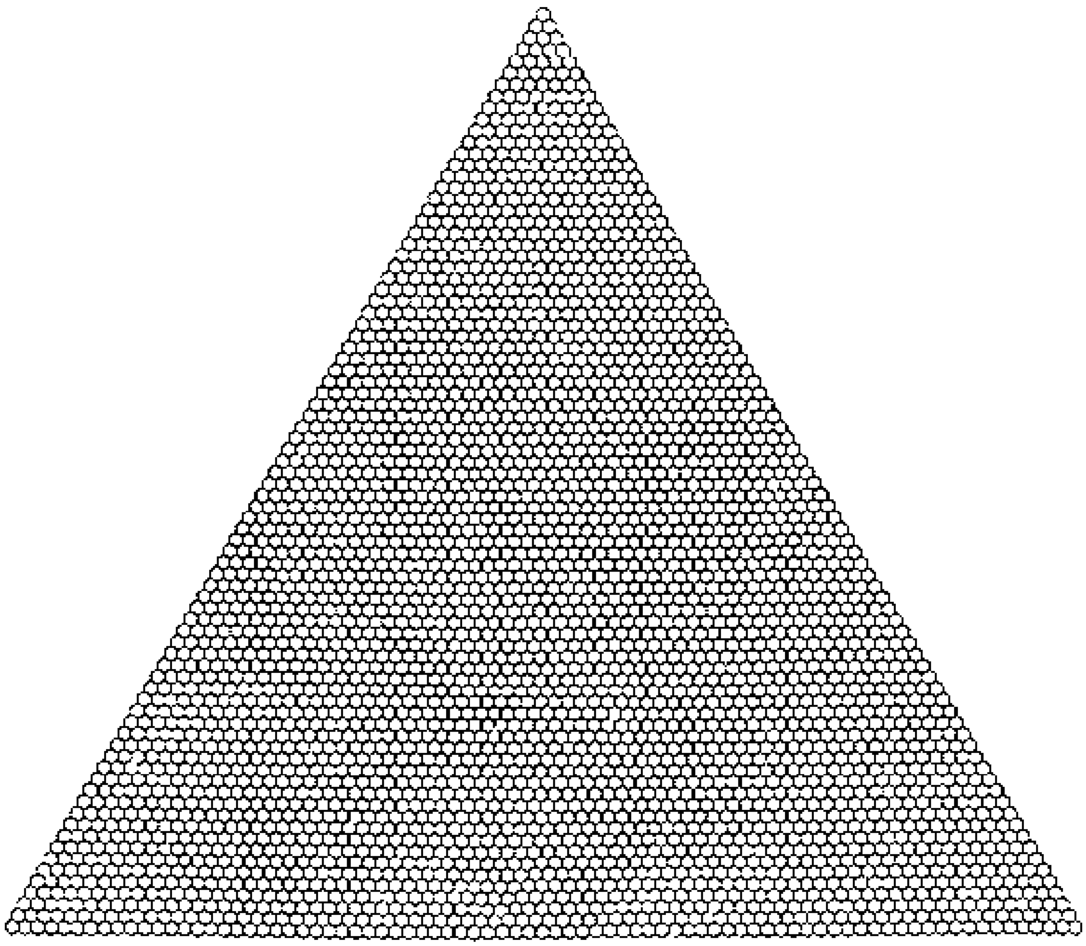
Transformation - includes the following actions:

Scaling -	enlarging or reducing the original figure
Rotation -	turning the figure through a certain angle
Translation -	sliding the figure to a new location

Student Resource #4

Optional Worksheet

Name: ____________________



Student Resource #5

Name________________

The Chaos Game

Start with one of the vertices.
Roll the die and move according to these rules:

For 1 or 2, move halfway to L
For 3 or 4, move halfway to T
For 5 or 6, move halfway to R.

Mark a dot wherever you land. Use this dot for your new starting place.
Continue playing until you have marked 30 dots.
Does this figure look familiar?

Teacher's Notes on The Chaos Game

To play the Chaos game, group students in pairs and give each pair a worksheet (Student Resource #5), a transparency sheet, a ruler, a die, and a transparency marker. You may wish to paper clip the transparency to the worksheet. Instruct students to follow the instructions on the worksheet. As the pairs finish, ask students to begin stacking transparencies together. Students should see the Sierpinski Triangle appear.

THE CHAOS GAME PROGRAM FOR THE TI-82 GRAPHING CALCULATOR

:ClrDraw	.....	clears the graphic screen
:0->Xmin	.....	these four commands set the viewing screen
:1->Xmax		to cover only from 0 to 1 on the *x*-axis
:0->Ymin		and from 0 to 1 on the *y*-axis
:1->Ymax		(remember our original triangle has sides of length one)
:0->C	.....	these three lines set *c*=0, *x*=.5, and *y*=0
:.5->X		*c* is just a counter so we know how many
:0->Y		points we have plotted while *x* and *y* are the coordinates of our initial point
:Lbl 1	.....	main program loop (labeled as 1)
:C+1->C		increase *c*, our counter, by 1 (*c*=*c*+1)
:PT-On(X,Y)		plot the point (*x*, *y*)
:If C>2000		check to see if 2000 points have been plotted
:Stop		if so, stop the program
:Rand->N		choose a random number between 0 and 1
:If N<.3333		if our random number is less than .3333 then
:Goto 2		go to section 2 of the program
:If N>.6666		if our random number is greater than .6666 then
:Goto 3		go to section 3 of the program
:.5X->X		if neither of these is true, then perform
:.5Y->Y		matrix operations for Copy 1
:Goto 1		go back to section 1 and do it again
:Lbl 2	.....	section 2 of program
:.5X+.5->X		perform matrix operations for Copy 2
:.5Y->Y		then go back to section 1 and do it again
:Goto 1		
:Lbl 3	.....	section 3 of program
:.5X+.25->X		perform matrix operations for Copy 3
:.5Y+.5->Y		then go back to section 1 and do it again
:Goto 1		

Student Resource #7

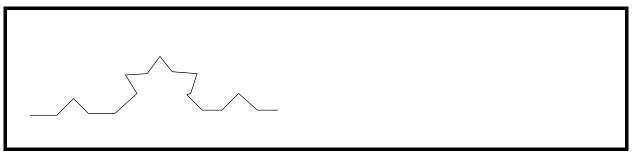
Worksheet 3

Name: ____________________

Koch Curve Activity

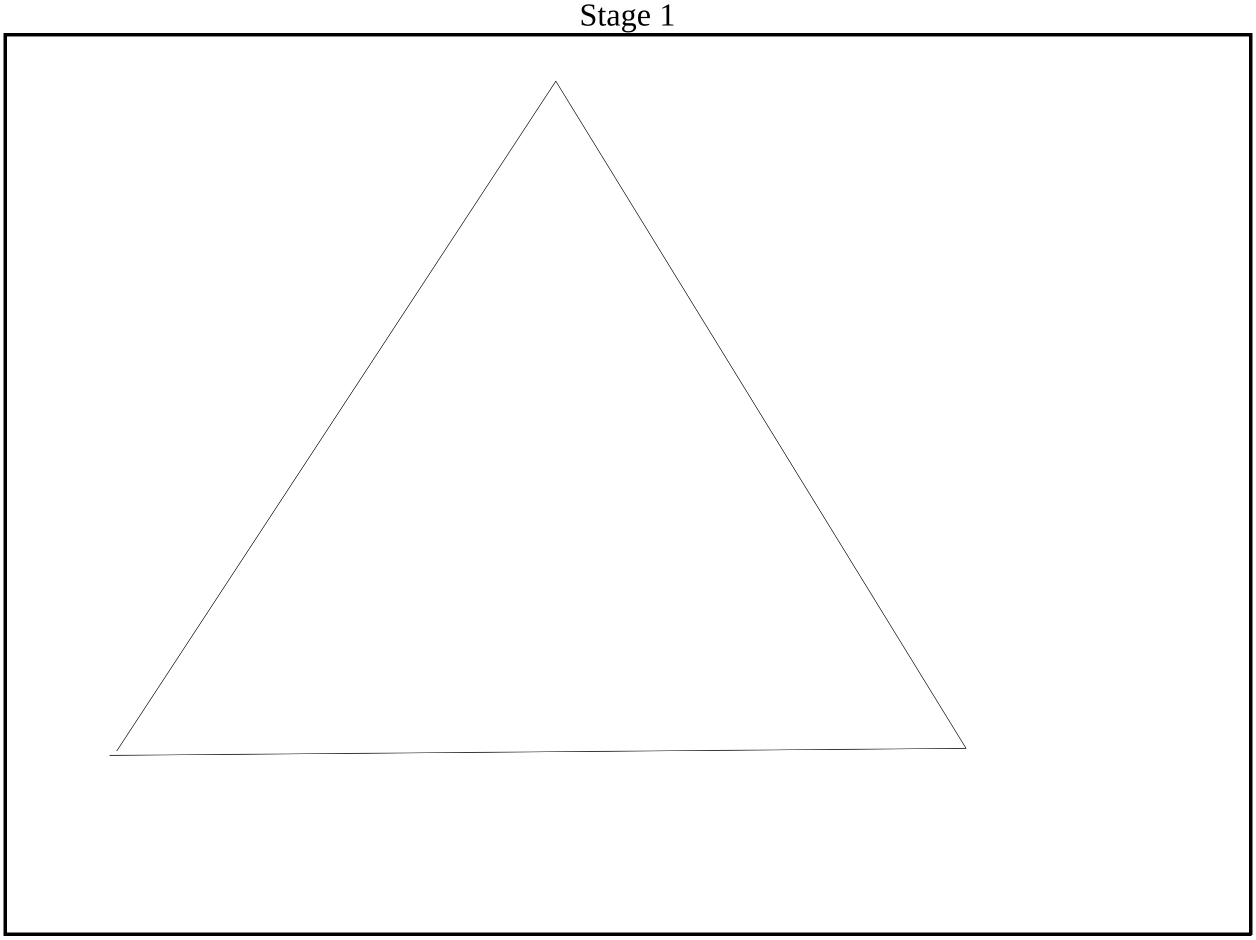
The Koch Curve is formed by removing the center third of a segment and replacing it by two equal segments to form a "tent," for example,

The same process is then repeated (iterated) on each of the segments.



Starting with this triangle, carry out the Koch Curve iterations to the fourth stage

Stage 1



Draw a circle around your figure. If you continued the Koch Curve process, would it ever cross the circle? If each of the original segments of the triangle were 27 units in length, what is the perimeter at stage 1?

at stage 2?

at stage 3?

at stage 4?

By what factor is the perimeter changing each time?

If you continued iterating, what would the perimeter be at stage 10?

at stage *n*?

Make a conjecture about the length of the Koch Curve at the infinite stage.

Worksheet 3
Koch Curve Activity
Answer Sheet

Draw a circle around your figure. If you continued the Koch Curve process, would it ever cross the circle? **No.** If each of the original segments of the triangle were 27 units in length, what is the perimeter at stage 1? **81**

at stage 2? **108**

at stage 3? **144**

at stage 4? **192**

By what factor is the perimeter changing each time? **4/3**

If you continued iterating, what would the perimeter be at stage 10? $\mathbf{81 * (4/3)^9 = 1078.78}$

at stage *n*? $\mathbf{81 * (4/3)^{(n-1)}}$

Make a conjecture about the length of the Koch Curve at the infinite stage. **The length of the Koch Curve at the infinite stage is infinity.**

Student Resource #8

Name: ____________________

Worksheet 4 (Optional)

MATRIX TRANSFORMATIONS

Objective: Calculate a transformation matrix, and use the resulting matrix equation to perform transformations on a rectangle.

The following transformations are to be performed on a rectangle:

1. scale the rectangle by a factor of ½.
2. rotate the rectangle 45 degrees counter-clockwise.
3. translate the rectangle 10 units to the right.
4. translate the rectangle 5 units down.

1. Calculate the values for the elements of the 2x2 matrix (this will perform the rotation and the scaling of the rectangle).

   a = s cos (R) = ____________

   b = ____________________ = ____________

   c = ____________________ = ____________

   d = ____________________ = ____________

2. Complete the blanks in the transformation matrix equation below:

$$\left|\begin{matrix} \quad & \quad \\ \quad & \quad \end{matrix}\right| \left|\begin{matrix} x \\ y \end{matrix}\right| + \left|\begin{matrix} \quad \\ \quad \end{matrix}\right| = \left|\begin{matrix} new\ x \\ new\ y \end{matrix}\right|$$

3. The rectangle that we wish to transform has coordinates (0,0), (20,0), (20,10), and (0,10). Graph this original rectangle on a sheet of graph paper.

4. Now use the matrix transformation equation that you wrote in Step 2 to transform each vertex into a new vertex. The original vertex would be (x, y) and the transformed vertex would be (new x, new y).

   Vertex 1 (0,0):

   Vertex 2 (20,0):

   Vertex 3 (20,10):

   Vertex 4 (0,10):

5. Now plot the four transformed vertices on the graph paper with the original rectangle. Connect the vertices to form the rectangle. How does the resulting rectangle compare to the original? Is this what you expected?

TEACHER NOTES: MATRIX TRANSFORMATIONS

Matrices can be used to perform TRANSFORMATIONS on geometrical figures or points. A transformation can include any or all of the following:

1. Scaling - enlarging or reducing the original figure
2. Rotation - turning the figure through a certain angle (Positive angles are measured in a counter-clockwise direction.)
3. Reflection - flipping the figure over
4. Translation - sliding the figure to a new location

To perform a transformation on a figure, we perform the matrix operations on each vertex. This results in a new set of vertices that define our transformed figure. The matrix operations are performed as follows:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} new\ x \\ new\ y \end{vmatrix}$$

where (x, y) is a vertex of the figure and (new x, new y) is the transformed vertex.

The elements of our transformation matrix are determined by the amount of scaling and the angle of rotation for our figure. Let s = scale factor and R = angle of rotation, then:

$a = s \cos(R)$
$b = -s \sin(R)$
$c = s \sin(R)$
$d = s \cos(R)$

EXAMPLE: Write the transformation matrix that would reduce a figure by a factor of 1/3, and rotate it 30 degrees counter-clockwise.

Solution:

Scaling factor: $s = .3333$
Angle of Rotation: $R = 30$

$a = .3333 \cos(30) = .2886$
$b = -.3333 \sin(30) = -.1667$
$c = .3333 \sin(30) = .1667$
$d = .3333 \cos(30) = .2886$

Therefore, the resulting transformation matrix is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} .2886 & -.1667 \\ .1667 & .2886 \end{vmatrix}$$

ONE additional factor that we want to add to our transformation is a translation, or sliding, of the figure. We can perform a translation by adding a 2x1 matrix with elements *e* and *f* as shown:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} e \\ f \end{vmatrix} = \begin{vmatrix} new\ x \\ new\ y \end{vmatrix}$$

where *e* will slide the figure *e* units in the *x*-direction,
and *f* will slide the figure *f* units in the *y*-direction.

Suppose, in addition to the scaling and rotation, that we wanted to move the figure in the previous example 5 units to the right and 3 units down. The following matrix equation would be used to transform the vertices:

$$\begin{vmatrix} .2886 & -.1667 \\ .1667 & .2886 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} 5 \\ -3 \end{vmatrix} = \begin{vmatrix} new\ x \\ new\ y \end{vmatrix}$$

SUMMARY:

1. The Matrix Transformation Equation is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} e \\ f \end{vmatrix} = \begin{vmatrix} new\ x \\ new\ y \end{vmatrix}$$

2. The values of the elements *a*, *b*, *c*, and *d* are calculated from the scale factor and the angle of rotation.

3. The values of the elements *e* and *f* are simply the translations in the *x*- and *y*-directions respectively.

4. You must perform the matrix multiplication and matrix addition to obtain a transformed coordinate for EACH vertex in the original figure. In other words, each vertex in the original figure (*x*, *y*), must undergo a matrix transformation which will yield a transformed vertex (new *x*, new *y*).

5. When you plot and connect the new vertices, they will form the transformed figure.

EXTENSION: MATRICES, FRACTALS, AND THE GRAPHING CALCULATOR

A fractal can be generated using a graphing calculator (or computer) by determining the necessary transformations. This can be a very exciting (as well as challenging) activity for students.

To determine the transformations, perform the following steps:

1. Choose a fractal shape that is strictly self-similar, in other words, one that is made up of smaller parts that are all similar to the whole. The example we will use is the Sierpinski Triangle.

2. Look carefully at the Sierpinski Triangle and notice that the original triangle is really made up of three smaller Sierpinski Triangles. For each of the smaller copies needed to make the original, we must come up with a transformation matrix. In this case, we must calculate three matrices.

3. Remember that a transformation matrix is simply a way of encoding the scaling, rotation, and translation of the figure. To make things easier, we will let our original Sierpinski Triangle be an equilateral triangle with sides equal to one. With this in mind, we must find the scale factor, rotation, angle, and translation of each of the three smaller copies. These values are listed below:

   Copy 1 (lower left copy):
   - scale factor = .5
   - rotation angle = 0 degrees
   - translation: *x*-dir = 0 *y*-dir = 0

   Copy 2 (lower right copy):
   - scale factor = .5
   - rotation angle = 0 degrees
   - translation: *x*-dir = .5 *y*-dir = 0

   Copy 3 (top middle copy):
   - scale factor = .5
   - rotation angle = 0 degrees
   - translation: *x*-dir = .25 *y*-dir = .5

   Notice that all three of the copies have a scale factor of .5; they are all one-half as tall and one-half as wide as the original. Also, none of the copies are rotated. When we reduce a figure, it always shrinks to the lower left position because the origin stays the same. Therefore, Copy 1 is finished; it ends up in the lower left position. Copy 2 will also end up in this position until we move it. Since our original triangle had sides of length one, we must slide Copy 2 one-half unit to the right. Similarly, Copy 3 needs to be moved one-fourth of a unit to the right and one-half unit up.

4. The next step is to calculate the actual matrices based on the values listed above for the three copies. The result is the following three transformation matrix equations:

Copy 1:

$$\begin{vmatrix} .5 & 0 \\ 0 & .5 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \end{vmatrix} = \begin{vmatrix} new\ x \\ new\ y \end{vmatrix}$$

Copy 2:

$$\begin{vmatrix} .5 & 0 \\ 0 & .5 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} .5 \\ 0 \end{vmatrix} = \begin{vmatrix} new\ x \\ new\ y \end{vmatrix}$$

Copy 3:

$$\begin{vmatrix} .5 & 0 \\ 0 & .5 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{vmatrix} .25 \\ .5 \end{vmatrix} = \begin{vmatrix} new\ x \\ new\ y \end{vmatrix}$$

5. Now we can use our matrices in the Chaos Game. In the Chaos Game, we start with any random point, (x, y), and plot it. Next, we randomly choose one of the transformation matrices and perform the matrix operations on that point to obtain a new point, ($x$new, $y$new). Plot the new point, randomly choose another matrix, perform the matrix operations, and we have yet another point. Plot this point and continue this process over and over. Amazingly, these ‘randomly’ plotted points will begin to form an image of the original object, in this case, the Sierpinski Triangle.

6. The following page has a listing of the Chaos Program written for the TI-82. The program follows the process outlined in Step 5 above. Explanations for each of the program lines are listed down the right hand side of the page.

Student Resource #11

Unit Evaluation

Please give all answers in complete sentences.

1. Did you enjoy this unit? Please explain.

2. Name and describe 3 concepts you learned in this unit.

3. If you could change any part of this unit, how would you alter it?

4. On which activity did you do your best work/learn the most?
Please explain.

5. Do you have any other comments?